Impact of Dependencies Between Input Variables in Structural Reliability Problems Under Incomplete Probability Information

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[4]Institut de Mathématique de Toulouse (IMT),
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The Flood Model[6]

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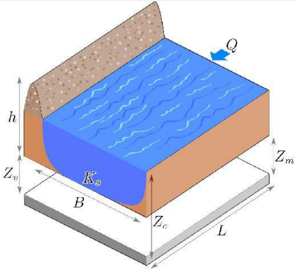
$$\text{with} \quad H = \left(\frac{Q}{BK_s\sqrt{\frac{Z_m - Z_v}{L}}} \right)$$



Input	Description	Probability Distribution
Q	Maximal annual flowrate	Truncated Gumbel $\mathcal{G}(1013, 558)$ on $[500, 3000]$
K_s	Strickler coefficient	Truncated normal $\mathcal{N}(30, 8)$ on $[15, +\infty]$
Z_v	River downstream level	Triangular $\mathcal{T}(49, 50, 51)$
Z_m	River upstream level	Triangular $\mathcal{T}(54, 55, 56)$
H_d	Dyke height	Uniform $\mathcal{U}[7, 9]$
C_b	Bank level	Triangular $\mathcal{T}(55, 55.5, 56)$
L	Length of the river stretch	Triangular $\mathcal{T}(4990, 5000, 5010)$
B	River width	Triangular $\mathcal{T}(295, 300, 305)$

[6] Bertrand Iooss and Paul Lemaître. "A review on global sensitivity analysis methods". In: *Uncertainty Management in Simulation-Optimization of Complex Systems*. Springer, 2015, pp. 101–122.

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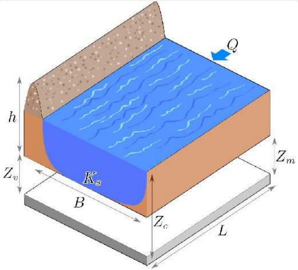

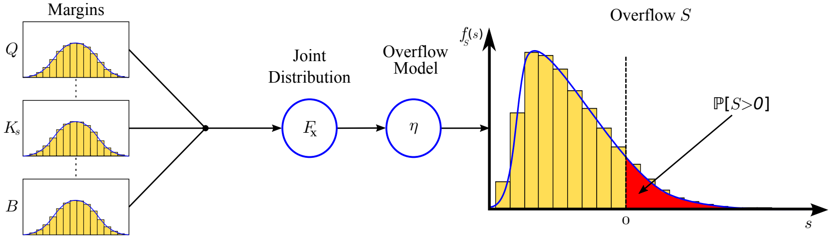

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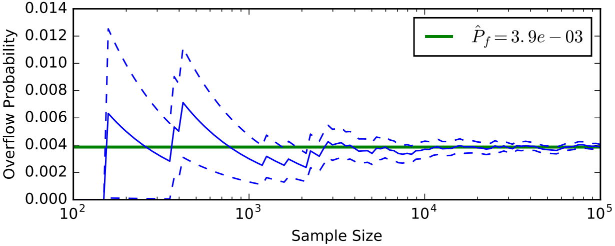
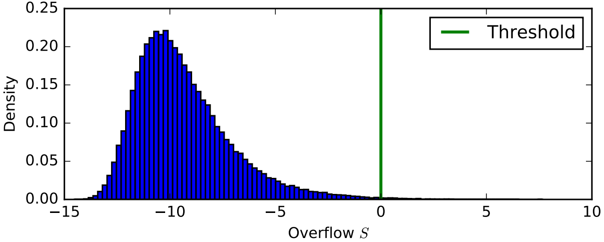
Monte-Carlo Estimation with the Independence Assumption

Estimation of $P_f = \mathbb{P}[S > 0]$.





Monte-Carlo Estimation with One Pair of Correlated Variables

We suppose that Q and K_s are correlated. How can this correlation impact $\hat{P}_f$?

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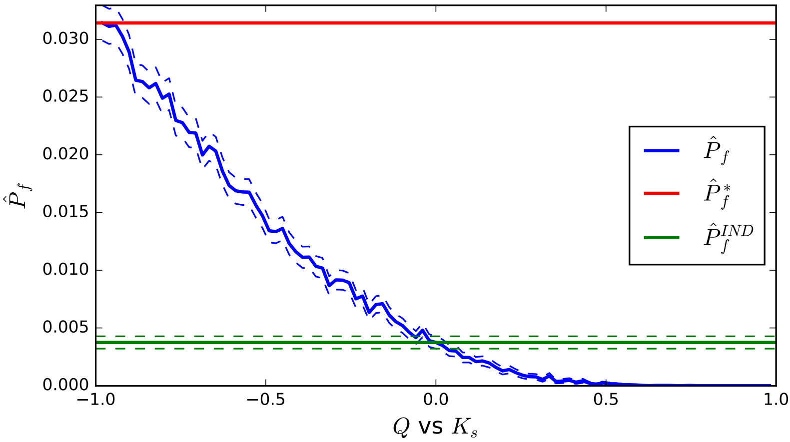

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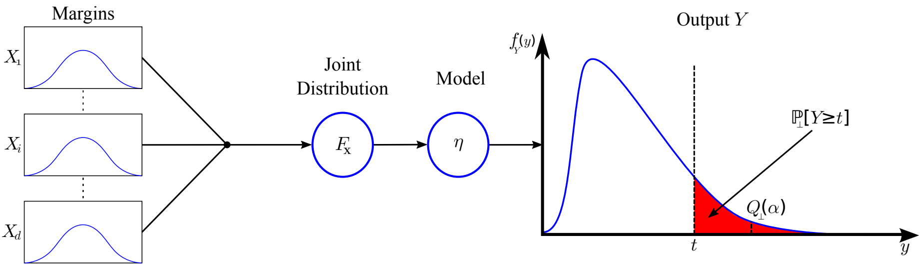
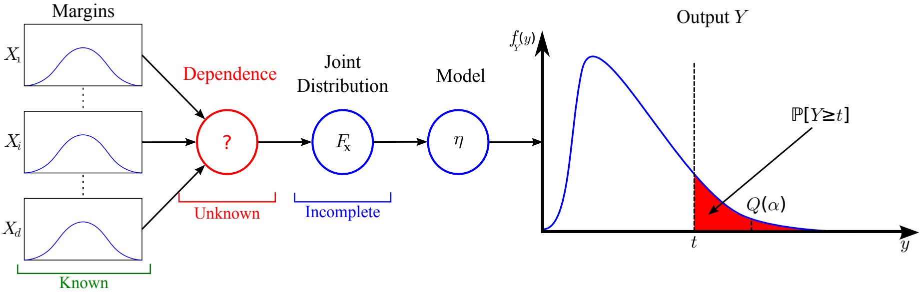

Margins
X_1
X_i
X_d
Joint Distribution
$F_{\mathbf{X}}$
Model
η
Output Y
$f_Y(y)$
$\mathbb{P}_{\perp}[Y \geq t]$
$Q_{\perp}(\alpha)$
t
y



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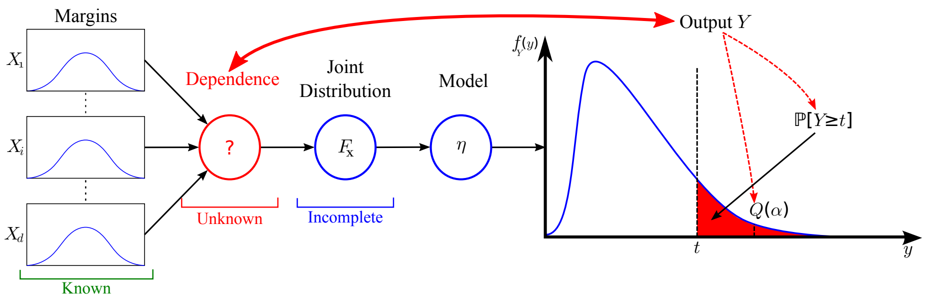
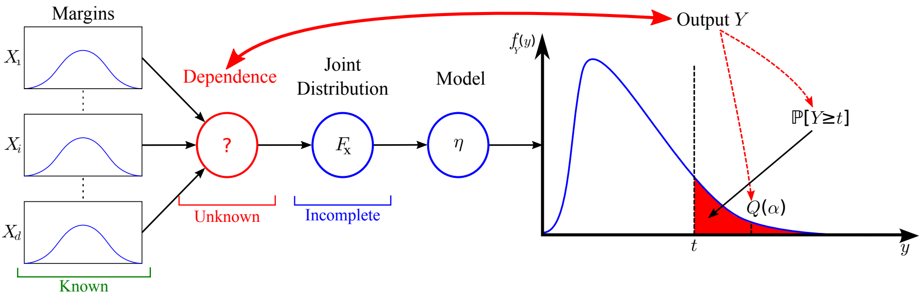


Questions:

- What is the relationship between the dependence structure and the output Y ?
- Is it conservative to suppose independence ?



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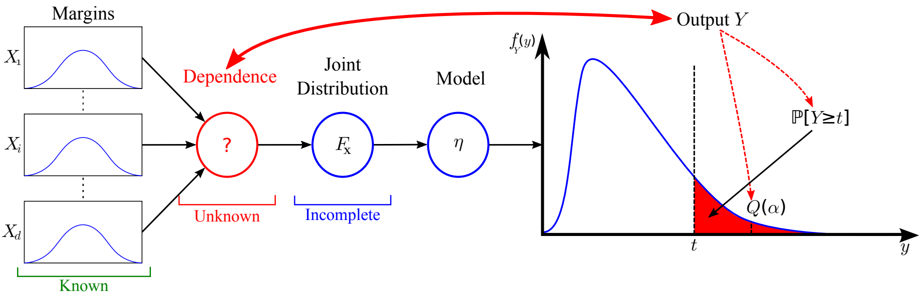
Objectives:

- Inform about the importance of the dependence structure.
- Bound the quantity of interest: e.g. $\mathbb{P}_{\perp}[Y \geq t] \leq \mathbb{P}^{*}[Y \leq t]$.
- Quantify the influence of the dependence for each pair of variables.



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How to describe the dependence structure?

Copulas

> *"Knowing the word "copula" as a grammatical term for a word or expression that links a subject and predicate, I felt that this would make an appropriate name for a function that links a multidimensional distribution to its one-dimensional margins"*
>
> *Sklar, 1996*

The dependence structure is described by a **parametric** copula $C_{\boldsymbol{\theta}}$ with $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^p$ such as[9,10]

$$F_{\mathbf{X}}(\mathbf{x}) = C_{\boldsymbol{\theta}}(F_{X_1}(x_1), \dots, F_{X_d}(x_d)).$$

Family	$C_\theta(u, v)$	Θ	Kendall's τ
Independent	uv	/	/
Gaussian	$\Phi_\theta(\Phi^{-1}(u), \Phi^{-1}(v))$	$[-1, 1]$	$\frac{2 \arcsin \theta}{\pi}$
Clayton	$(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$	$[0, \infty)$	$\frac{\theta}{2+\theta}$
Gumbel	$\exp\left\{-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}\right\}$	$[1, \infty)$	$1 - \frac{1}{\theta}$

[9] Roger B Nelsen. *An introduction to copulas*. Springer Science & Business Media, 2007.
[10] M Sklar. *Fonctions de répartition à n dimensions et leurs marges*. Institut de Statistique de l'Université de Paris, 1959.

Copulas

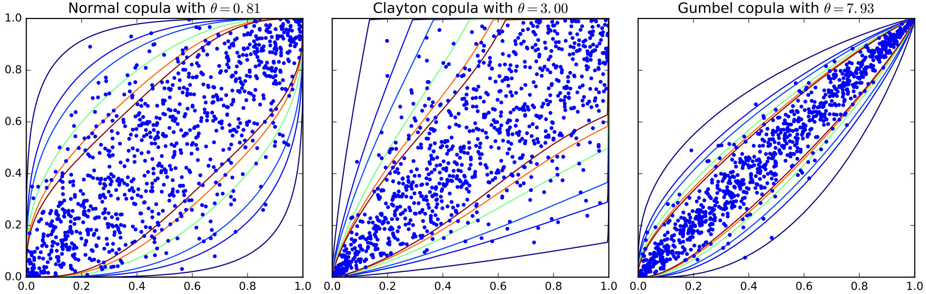


Figure: Example of copula densities with $\tau = 0.6$.

Copulas

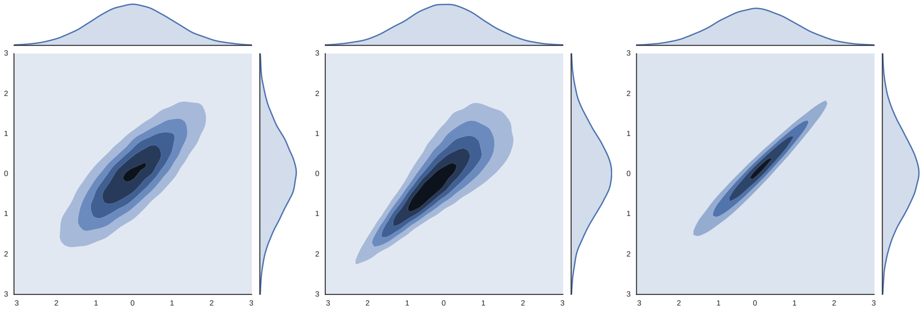
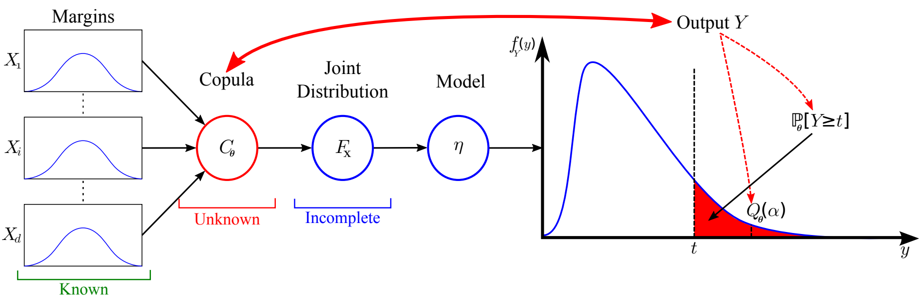


Figure: Example of joints p.d.f with Gaussian margins and $\tau = 0.6$.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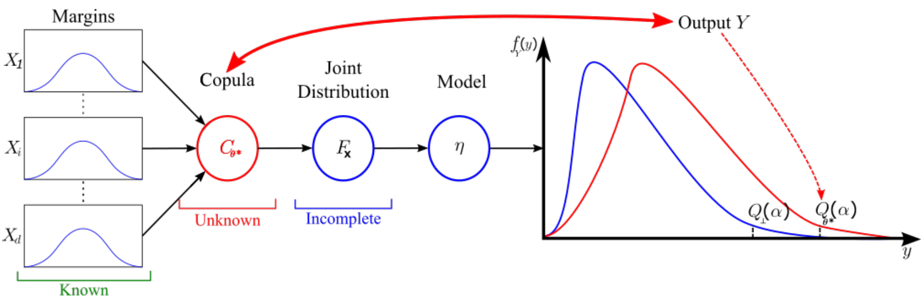
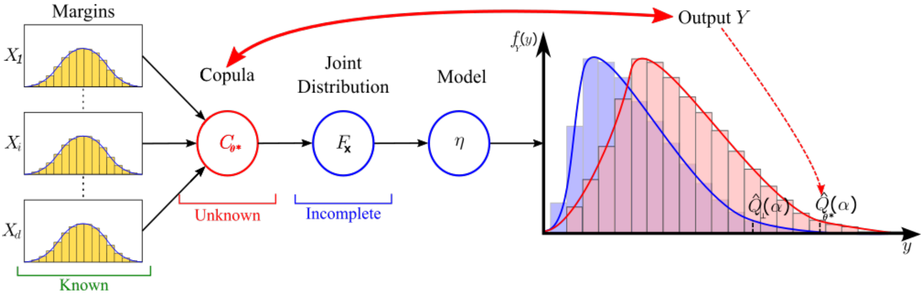


For a given $\alpha \in (0,1)$ and a copula $C_{\boldsymbol{\theta}}$,

$$\boldsymbol{\theta}^* = \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmax}}\, Q_{\boldsymbol{\theta}}(\alpha).$$

This worst case gives an upper bound such that

$$Q_{\boldsymbol{\theta}^*}(\alpha) \geq Q_{\perp}(\alpha).$$



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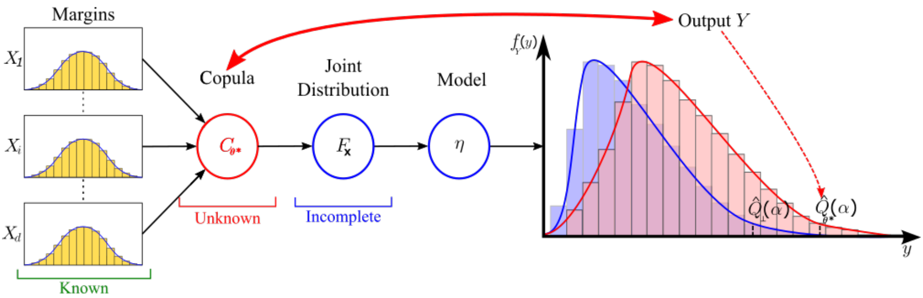

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In practice, $\boldsymbol{\theta}$ is discretized using a thin grid $\boldsymbol{\Theta}_K$ of cardinality K:

$$\hat{\boldsymbol{\theta}}_{n,K} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}_K}{\operatorname{argmax}}\, \hat{Q}_{n,\boldsymbol{\theta}}(\alpha).$$

The maximisation of a data-dependent function is an *extremum-estimator*[2,5]:

$$\hat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmax}} \hat{Q}_{n,\boldsymbol{\theta}}(\alpha).$$

Theorem 1 (Consistency of $\hat{\boldsymbol{\theta}}_n$)

If there is a function $Q_{\boldsymbol{\theta}}(\alpha)$, for any $\alpha \in (0, 1)$ such that

- $Q_{\boldsymbol{\theta}}(\alpha)$ *is uniquely minimised at* $\boldsymbol{\theta}^*$,
- Θ *is compact,*
- $Q_{\boldsymbol{\theta}}(\alpha)$ *is continuous in* $\boldsymbol{\theta}$,
- $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} |\hat{Q}_{n,\boldsymbol{\theta}}(\alpha) - Q_{\boldsymbol{\theta}}(\alpha)| \xrightarrow{P} 0$,

then $\hat{\boldsymbol{\theta}}_n \xrightarrow{P} \boldsymbol{\theta}^*$.

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- *$Q_{\boldsymbol{\theta}}(\alpha)$ is continuous in $\boldsymbol{\theta}$, ($\rightarrow$ OK under regularity assumptions of η and $F_{\mathbf{X}}$)*
- *$\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} |\hat{Q}_{n,\boldsymbol{\theta}}(\alpha) - Q_{\boldsymbol{\theta}}(\alpha)| \xrightarrow{P} 0$, ($\rightarrow$ OK, thanks to the DKW inequality)*

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Multivariate Gaussian Copula

The problem can be treated assuming a Gaussian copula with correlation matrix $\mathbf{R} \in [-1, 1]^{d \times d}$.
Example for $d = 3$:

$$\mathbf{R} = \begin{pmatrix} 1 & \theta_{12} & \theta_{13} \\ \theta_{12} & 1 & \theta_{23} \\ \theta_{13} & \theta_{23} & 1 \end{pmatrix}.$$

Objective: Determine the correlation matrix maximising the quantile.

[3] Paul Embrechts, Alexander McNeil, and Daniel Straumann. "Correlation and dependence in risk management: properties and pitfalls". In: *Risk management: value at risk and beyond* (2002), pp. 176–223.

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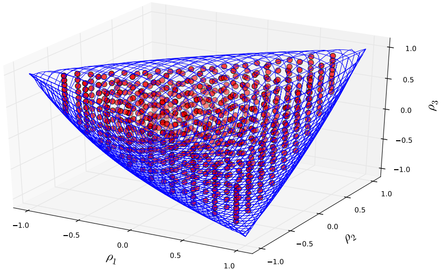

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- The correlation matrix $\mathbf{R}$ should be definite semi-positive.
- The Gaussian assumption is not always adapted.

> “ *fallacies raised from the naive assumption that dependence properties of the elliptical world also hold in the non-elliptical world*[3] ”

- The Gaussian copula does not have tail dependence: less penalizing.

[3] Paul Embrechts, Alexander McNeil, and Daniel Straumann. “Correlation and dependence in risk management: properties and pitfalls”. In: *Risk management: value at risk and beyond* (2002), pp. 176–223.

Two-Dimensional Problems

Several studies were made to study the influence of correlations[4] and using copulas[11,12].

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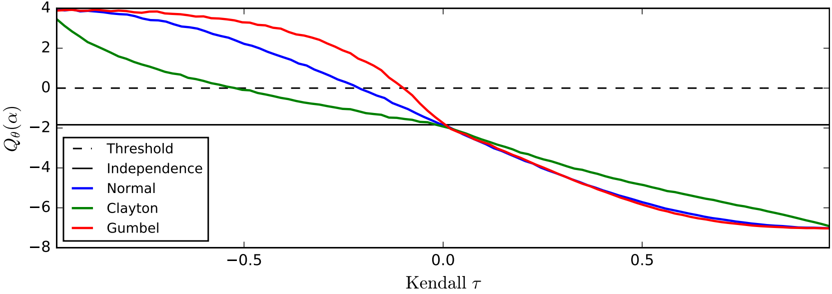
On the Flood Example with Q and K_s at $\alpha = 99\%$:



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Two-Dimensional Problems

The worst case is not always at the edge.
For example, we consider:

- $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(-2,1)$
- $\eta(x_1, x_2) = x_1^2 x_2^2 - 0.3 x_1 x_2$

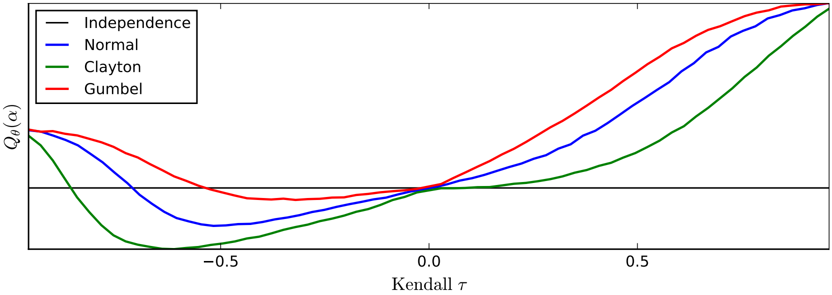

Figure: Variation of the output quantile for different copula families and $\alpha = 5\%$.

Copula Densities

Using Sklar's Theorem, a bivariate distribution of X_1 and X_2 can be written using a copula C_{12}, such that[8]

$$\mathbf{F}(x_1, x_2) = C_{12}(F_1(x_1), F_2(x_2)).$$

If F_1 and F_2 are continuous, C is unique and admits a density

$$c_{12}(u_1, u_2) = \frac{\partial^2 C_{12}(u_1, u_2)}{\partial u_1 u_2}.$$

Which implies

- joint density:

$$f(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_2) \cdot f_2(x_2)$$

- conditional density:

$$f(x_1 | x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)$$

[8] Nicole Krämer and Ulf Schepsmeier. "Introduction to vine copulas". In: ().

Pair-Copula Construction (R-Vines)

The joint density $f(x_1, \ldots, x_d)$ can be represented by a product of pair-copula densities and marginal densities[7].

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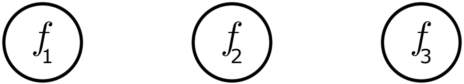

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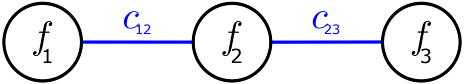
$$\times\, c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3))_{\text{(unconditional pairs)}}$$



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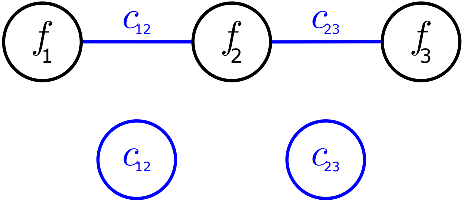

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& \times c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2))_{\text{(conditional pair)}}
\end{aligned}
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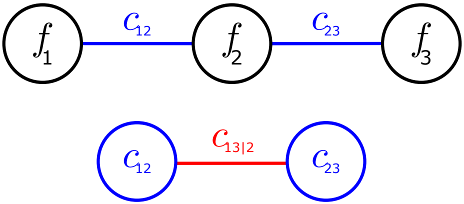

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Vine Copulas

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- Graphical model: set of connected trees.

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- Multidimensional dependence structure using bivariate copulas from various families.
- Graphical model: set of connected trees.

Problem: There is very large number of possible Vine decompositions :
$\binom{d}{2} \times (n-2)! \times 2^{\binom{d-2}{2}}$.

For example, when $d = 6$, there are 23.040 possible R-Vines.

Iterative Vine Construction

Key point: Not all pairs are equivalently influential on $Q_{\boldsymbol{\theta}}(\alpha)$.

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Idea: Iteratively add a pair in the maximisation of $Q_{\boldsymbol{\theta}}(\alpha)$.

Init: $\boldsymbol{\Theta}^c = \emptyset$
1: For each pair X_i-X_j:

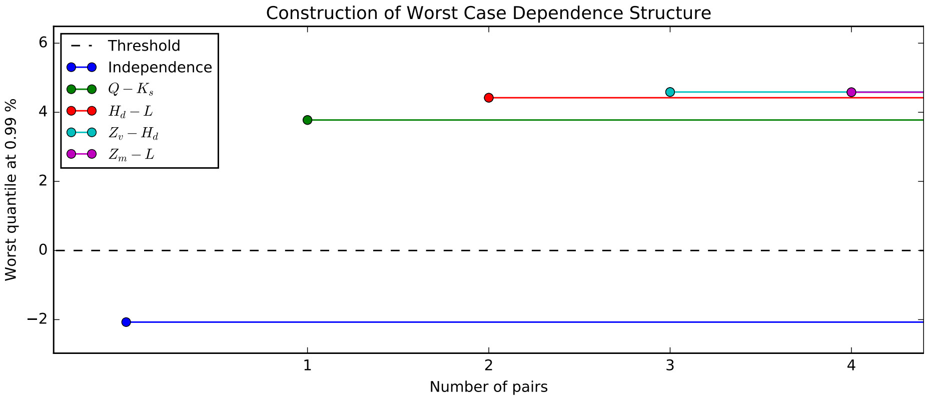
$$\hat{\theta}^*_{ij} = \underset{\boldsymbol{\theta}_{ij} \in \boldsymbol{\Theta}_{ij} \cup \boldsymbol{\Theta}^c}{\operatorname{argmax}} \hat{Q}_{n,\boldsymbol{\theta}_{i,j}}(\alpha),$$

2: Add the pair X_i-X_j that maximises the most $Q_{\boldsymbol{\theta}}(\alpha)$ in $\boldsymbol{\Theta}^c$. And adapt the R-Vine structure.
3: Loop over **2** and **3**. Stop when there is no evolution of $Q_{\boldsymbol{\theta}^*}(\alpha)$ or when the budget is consumed.

Iterative Vine Construction

Key point: Not all pairs are equivalently influential on $Q_{\boldsymbol{\theta}}(\alpha)$.

Idea: Iteratively add a pair in the maximisation of $Q_{\boldsymbol{\theta}}(\alpha)$.



Contents

Conclusion

- Dependencies can have an significant impact on the model output Y.
- The worst case scenario $\boldsymbol{\theta}^*$ gives an idea of the influence of dependencies.
- Using Vine Copulas we can determine a penalized dependence structure.
- Structured methodology with a discretized $\boldsymbol{\Theta}$, parametric copula families and Vines.

Conclusion

- Dependencies can have an significant impact on the model output Y.
- The worst case scenario $\boldsymbol{\theta}^*$ gives an idea of the influence of dependencies.
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Perspectives

- Aggregate expert feedback to restrict the set $\boldsymbol{\Theta}$ to more realistic dependence structures.
- Apply a more greedy algorithm using Quantile Regression Forests.

Thank You!

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How to measure the impact of potential dependencies?

Definition 1 (Price of Correlation,[1])

[a] Shipra Agrawal et al. "Price of correlations in stochastic optimization". In: *Operations Research* 60.1 (2012), pp. 150–162.

Given a decision $\mathbf{z} \in \mathcal{Z}$*, a collection of joint densities* $\mathcal{X}$ *and a cost function* h*,*

$$\eta(\mathbf{z}) = \sup_{\mathbf{X} \in \mathcal{X}} \mathbb{E}_{\mathbf{X}}[h(\mathbf{X}, \mathbf{z})].$$

Let $\mathbf{z}_I = \text{argmin}_{\mathbf{z} \in \mathcal{Z}} \mathbb{E}_{\mathbf{X}_I}[h(\mathbf{X}_I, \mathbf{z})]$*,* $\mathbf{z}_R = \text{argmin}_{\mathbf{z} \in \mathcal{Z}} \eta(\mathbf{z})$*. Then* ***Price of Correlation (POC)*** *is defined as:*

$$\mathrm{POC} = \frac{\eta(\mathbf{z}_I)}{\eta(\mathbf{z}_R)}$$

The application is different, but a common question remains: how much risk it involves to ignore the correlations?

To estimate the influence of a pair of variables X_i-X_j on $Q_{\boldsymbol{\theta}}(\alpha)$, we define the quantity

$$\mathscr{I}_{ij} = \frac{|Q_{\theta^*_{ij}}(\alpha) - Q_\perp(\alpha)|}{|Q^*_{\boldsymbol{\theta}}(\alpha) - Q_\perp(\alpha)|},$$

where θ^*_{ij} is the solution of the max(min)imisation of $Q_{\boldsymbol{\theta}}(\alpha)$ when only the pair X_i-X_j is correlated and the others are considered independent.

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